

LOW ENERGY CLAMPED PLANAR ELASTICA

LYLE NOAKES

1. CONTINUOUS ELASTICA

The Euclidean plane E^2 be the Euclidean plane, equipped with the Euclidean inner product $\langle \cdot, \cdot \rangle$, is identified with the complex plane \mathbb{C} in the standard way, with $(0, 1)$ corresponding to \mathbf{i} . The unit circle in E^2 is denoted by S^1 , and we suppose that real numbers $a < b$ are given. Given $x_a, x_b \in E^2$, together with $v_a, v_b \in S^1 \subset E^2$, suppose that there exists a C^∞ unit-speed curve $x : [a, b] \rightarrow E^{q+1}$ satisfying

$$(1) \quad x(a) = x_a, \quad \dot{x}(a) = v_a, \quad x(b) = x_b, \quad \dot{x}(b) = v_b.$$

Such a curve has length $L := b - a$, and is said to be *feasible* for a, b, x_a, x_b, v_a, v_b . An Euler-Bernoulli (fixed length) *elastica* is defined to be a critical point of the elastic energy functional

$$E(x) := \frac{1}{2} \int_a^b \|\ddot{x}(t)\|^2 dt$$

as x varies over feasible curves, where $\|\cdot\|$ is the Euclidean norm. The beautiful review [5] of the study of elastica, from James Bernoulli and Leonhard Euler through to 2008, contains many interesting references including [1]. As explained in [5], the difficulty of obtaining numerical solutions for elastica satisfying prescribed conditions was influential in the development of the modern theory of splines. The present paper attempts a small additional contribution, by way of simplicity and speed, to the well studied area of numerical methods for elastica [4], [2].

From the Pontryagin Maximum Principle [6], a feasible x is an elastica when, for some C^∞ function $\mu : [a, b] \rightarrow \mathbb{R}$,

$$(2) \quad \ddot{v}(t) + \mu(t)v(t) = C$$

where $v(t) := \dot{x}(t)$, and $C \in E^{q+1}$ is constant. Taking inner products with $v(t) \in S^1$, we see that

$$\mu(t) = \langle C, v(t) \rangle - \langle \ddot{v}, v \rangle = \langle C, v(t) \rangle + \kappa(t)^2$$

where *curvature* is defined by $\kappa(t) := \det[v(t)|\dot{v}(t)] = \langle v(t), \dot{\mathbf{i}}v(t) \rangle$.

Definition 1. A lifting of $v : [a, b] \rightarrow S^1$ is a C^∞ function $\theta : [a, b] \rightarrow \mathbb{R}$ satisfying $(\cos \theta(t), \sin \theta(t)) = v(t)$ for all $t \in [a, b]$. \square

For any lifting θ of v , $\kappa = \dot{\theta} = \pm \|\dot{v}\|$. Taking inner products of (2) with $\dot{v}(t)$,

$$\langle \ddot{v}(t), \dot{v}(t) \rangle = \langle C, \dot{v}(t) \rangle \implies \langle C, v(t) \rangle = \frac{\kappa(t)^2 - c}{2} \implies \mu(t) = \frac{3\kappa(t)^2 - c}{2},$$

where $c \in \mathbb{R}$ is constant. Differentiating (2) and taking inner products with $\dot{v}(t)$,

$$0 = \langle v^{(3)}, \dot{v} \rangle + \mu\kappa^2 = \kappa\ddot{\kappa} + \dot{\kappa}^2 - \langle \ddot{v}, \dot{v} \rangle + \mu\kappa^2 = \frac{\kappa(t)}{2}(2\ddot{\kappa}(t) + \kappa^3(t) - c\kappa(t))$$

because $\ddot{v} = -\kappa^2 v + \dot{\kappa}\mathbf{i}v$. Therefore (or, alternatively, following the derivation in [7]),

$$(3) \quad 2\ddot{\kappa}(t) = c\kappa(t) - \kappa(t)^3.$$

Excluding the trivial cases where κ is constant, namely x is either a circular arc or a line segment, the solutions of (3) are

$$(4) \quad \kappa(t)^2 = \kappa_0^2 \left(1 - \frac{p^2}{w^2} \operatorname{sn}^2 \left(\frac{\kappa_0}{2w} (t - t_0), p \right) \right)$$

where sn denotes the *elliptic sine*, w is either p or 1, and c is related to the parameters κ_0, p by

$$2c = \frac{\kappa_0^2}{w^2} (3w^2 - p^2 - 1).$$

For $w = p = 1$ the elastica is called *borderline*. Otherwise, according as $w = p$ or $w = 1$, it is said to be *wavelike* or *orbitlike*.

2. NONTRIVIAL CASES

If x is an elastica then, for any Euclidean transformation A of E^2 , so is $t \mapsto Ax(t)$. So suppose, without loss, that $x_a = (0, 0)$ and $v_a = (1, 0)$.

Date: August 17, 2016.

2.1. Wavelike. For a wavelike elastica, κ oscillates periodically between $\pm\kappa_0$, according to

$$(5) \quad \kappa(t) = \kappa_0 \operatorname{cn}\left(\frac{\kappa_0}{2p}(t - t_0), p\right),$$

where the *elliptic cosine* cn is given by $\operatorname{cn}(u, p) = \cos \phi$ where ϕ is the *Jacobi amplitude* $\operatorname{am}(u, p)$, namely

$$u = F(\phi, p) := \int_0^\phi \frac{1}{\sqrt{1 - p \sin^2 \psi}} d\psi.$$

2.2. Orbitlike. For an orbitlike elastica we have

$$(6) \quad \kappa(t) = \kappa_0 \operatorname{dn}\left(\frac{\kappa_0}{2}(t - t_0), p\right),$$

where $\operatorname{dn}(u, p) := \sqrt{1 - p \sin^2 \phi}$, with $\phi = \operatorname{am}(u, p)$ as before. Integrating (6),

$$\theta(t) = 2 \left(\operatorname{am}\left(\frac{\kappa_0(t - t_0)}{2}, p\right) - \operatorname{am}\left(\frac{\kappa_0(a - t_0)}{2}, p\right) \right),$$

where $\theta : [a, b] \rightarrow \mathbb{R}$ is the lifting of v with $\theta(a) = 0$.

Example 1. Taking $a = 0$, $b = 10$, $\kappa_0 = 1$, $t_0 = 1/2$ and $p = 2$, we find that $\theta(t) = 2\operatorname{am}(t/2 - 1/4, 2) + 0.489774$. The corresponding elastica $x : [0, 10] \rightarrow E^2$, shown in Figure 1, is found by numerically solving $\dot{x}(t) = (\cos \theta(t), \sin \theta(t))$. Mathematica's *NDSolve* takes 0.586 seconds on a 1.7GHz Intel Core i5 Mac with 4GB RAM. \square

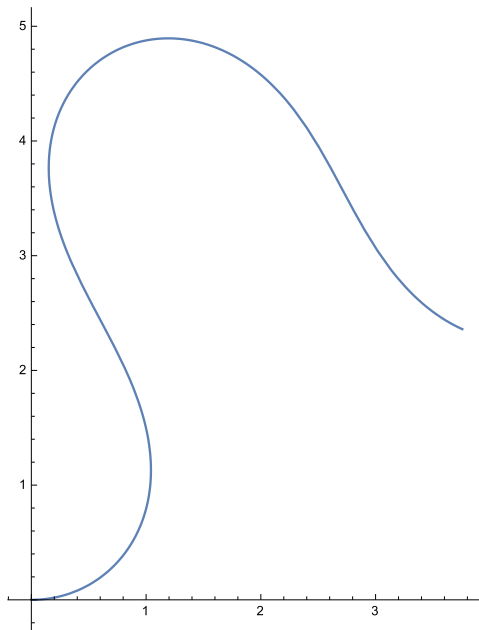


FIGURE 1. An Orbitlike Elastica (Example 1).

2.3. Borderline. For a borderline elastica, $\kappa(t) = \kappa_0 \operatorname{sech}(\frac{\kappa_0}{2}(t - t_0))$ is nonperiodic.

3. BOUNDARY CONDITIONS

Given a, b, x_a, v_a , the elastica x is determined by its curvature $\kappa : [a, b] \rightarrow \mathbb{R}$. So the parameters κ_0, p, t_0 must be chosen to satisfy $x(b) = x_b$ and $\dot{x}(b) = v_b$. This can be done, separately for each case, by numerically solving a system of three nonlinear equations, usually in elliptic functions. This is time-consuming and there are generally multiple solutions¹ to the boundary value problem. With applications and extensions in mind, we make it a point to search for uncomplicated clamped elastica, especially *minimisers* of E .

Background on finding elastic splines² can be found in §16 of [5], with more details in [4]. More recently, a contribution by Bruckstein, Holta, Netravalia and Arun[2] solves boundary value and interior value problems of this sort, and in much greater generality than considered in the present paper. Their method, which we call the *standard discretisation*, proceeds by optimising a discrete analogue of energy for piecewise-linear curves satisfying the given constraints. One of the great advantages of standard discretisation is ease of implementation, but the method can be time-consuming and may easily result in clamped elastica of unnecessarily high energies.

Example 2. As in Example 1, take $a = 0$, $b = 10$, $x_a = (0, 0)$ and $v_a = (1, 0)$. Then (cheating a little) take $x_b = x(b) = (3.75605, 2.35942)$ and $v_b = \dot{x}(b) = (0.911711, -0.410832)$ where x is the elastica found in Example 1. Subdividing $[0, 10]$ into 100 subintervals, the vertices of the corresponding discrete elastica are shown (red) in Figure 2, together with the original x (blue). Standard discretisation takes 597 seconds to find the discrete elastica, which has discrete energy 0.540 compared with

¹ This contrasts with the analogous problem where x is not necessarily unit-speed and the solution is a unique cubic polynomial.

² A different problem, where L is not considered in advance, is studied by Brunnett and Wendt [3].

0.043 for 101 equally spaced points along x . Although the standard discretisation is straightforward, some appreciable effort is needed to compute it, and the discrete energy is much too high. \square

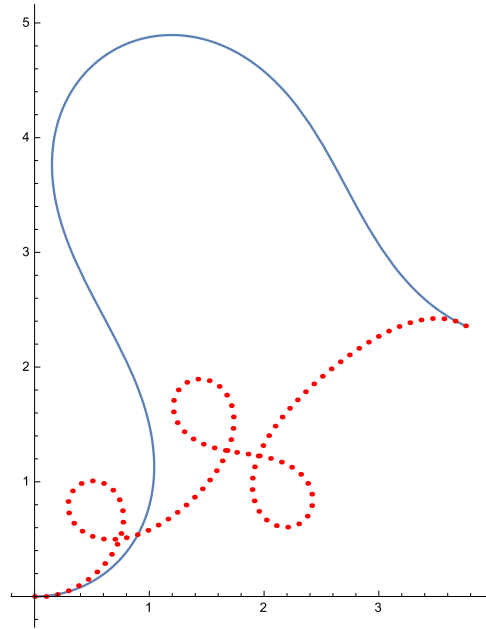


FIGURE 2. Continuous (blue) and standard discretisation (red) equal-length elastica, with the same length and end conditions (Examples 2, 4).

Standard discretisation seems a big improvement over classical methods, but cases such as Example 2 suggest that it may be insufficiently robust for routine use by nonexperts interested in uncomplicated curves.

Our method for estimating clamped elastica is designed to search for elastica x of small energy. Indeed, in the first step, we make the stronger assumption that all derivatives of v are moderate in size. Then the assumption is used to find a first estimate $\tilde{\theta}$ of a lifting θ of the unknown elastica x . In the second step, $\tilde{\theta}$ is taken as a starting point for a numerical optimisation of an approximate energy. Because $\tilde{\theta}$ is already uncomplicated, and reducing energy should not make things worse, we are more likely to achieve a global minimum of E , rather than just a local minimum.

Computational speed is addressed in Step 2, where Simpson's Rule improves approximations to E , and estimates of θ are modelled as polynomial splines. In effect, smoothness of elastica is used to reduce the need for a large search space. Numerical optimisation then proceeds quickly.

4. STEP 1

Minimising $E(x)$ means minimising the L^2 norm of \dot{v} . In this first step we aim for an uncomplicated initial curve \tilde{x} where the L^2 norms of derivatives of all orders are not too large, having regard to $L = b - a$ and approximately satisfying the prescribed boundary conditions. Rather than construct \tilde{x} explicitly, we estimate $\tilde{v} := \dot{\tilde{x}}$, then a lifting $\tilde{\theta} : [a, b] \rightarrow \mathbb{R}$ of \tilde{v} . Then $\tilde{\theta}$ is used in Step 2, to start the optimisation of an approximation to E .

Definition 2. Writing $h := (b - a)/4$, a quantity f calculated from x is said to be $O(h^n)$ when, for some constant K depending only on some class³ to which x belongs (not specifically on x), the magnitude of f is bounded above by Kh^n . So a vector valued function f calculated from x and defined on $[a, b]$ is $O(h^n)$ when, for some constant K independent of x and all $t \in [a, b]$, we have $\|f(t)\| \leq Kh^n$. \square

Assuming $v^{(m)} = O(1)$ for $1 \leq m \leq 4$, write $v_k := v(a + kh)$ for $0 \leq k \leq 4$ so that, approximating $v^{(4)}$ by central differences,

$$\frac{v_4 - 4v_3 + 6v_2 - 4v_1 + v_0}{h^4} = v^{(4)}(a + 2h) + O(h^2) = O(1) \implies$$

$$(7) \quad -2(v_1 + v_3) + 3v_2 = -\frac{1}{2}(v_a + v_b) + O(h^4).$$

From $\int_a^b v(t) dt = x_b - x_a$ we find, using⁴ the Composite Simpson's Rule,

$$(8) \quad 2(v_1 + v_3) + v_2 = \frac{3}{2h}(x_b - x_a) - \frac{1}{2}(v_a + v_b) + O(h^4).$$

Eliminating $v_1 + v_3$ between (7), (8), we find that $v_2 = \tilde{w}_2 + O(h^4)$ where

$$(9) \quad \tilde{w}_2 := \frac{3}{8h}(x_b - x_a) - \frac{1}{4}(v_a + v_b).$$

³Here a and b may vary, depending on the choice of x within the class.

⁴Boole's Rule might be used instead, or (with additional complexity) Gauss-Lobatto quadrature, but there is a lot to be said for simplicity.

Therefore, and because $\|v_2\| = 1$, we have $v_2 = \tilde{v}_2 + O(h^4)$, where $\tilde{v}_2 := \tilde{w}_2/\|\tilde{w}_2\|$.

Substituting for v_2 in (8), we find that $2w_{1,3} := v_1 + v_3 = 2\tilde{w}_{1,3} + O(h^4)$ where

$$(10) \quad \tilde{w}_{1,3} := \frac{3}{8h}(x_b - x_a) - \frac{1}{8}(v_a + v_b) - \frac{1}{4}\tilde{v}_2.$$

By Taylor's Formula⁵,

$$(11) \quad v_3 - v_1 = \frac{1}{2}(v_b - v_a) + O(h^3),$$

so that $v_1 = \tilde{w}_1 + O(h^3)$ and $v_3 = \tilde{w}_3 + O(h^3)$ where

$$(12) \quad \tilde{w}_1 := \tilde{w}_{1,3} - \frac{1}{4}(v_b - v_a) \quad \text{and} \quad \tilde{w}_3 := \tilde{w}_{1,3} + \frac{1}{4}(v_b - v_a).$$

Then, because v_1 and v_3 are unit vectors, $v_1 = \tilde{v}_1 + O(h^3)$ and $v_3 = \tilde{v}_3 + O(h^3)$ where $\tilde{v}_j := \tilde{w}_j/\|\tilde{w}_j\|$. So we have estimated v_1 and v_3 to $O(h^3)$ errors, and v_2 to $O(h^4)$. Summarising so far,

Proposition 1. *Given $a < b \in \mathbb{R}$, $x_a, x_b \in E^2$, $v_a, v_b \in S^1$, define $\tilde{v}_j := w_j/\|w_j\| \in S^1$ for $j = 1, 2, 3$, where w_2 is given by formula (9), and w_1, w_3 by (12). Then for suitably small h , and assuming that derivatives of x are $O(1)$, we have $v_2 = \tilde{v}_2 + O(h^4)$ and, for $j = 1, 3$, $v_j = \tilde{v}_j + O(h^3)$. \square*

Next the \tilde{v}_j are used to find a rough estimate $\tilde{\theta}$ of the lifting $\theta : [a, b] \rightarrow \mathbb{R}$ of v where, without loss of generality, $v_a = (1, 0)$ with $\theta(a) = 0$. We write $\tilde{v}_4 := v_b$, $\tilde{\theta}_0 := \tilde{\theta}(a) = 0$ and require

$$(13) \quad (\cos \tilde{\theta}(a + jh), \sin \tilde{\theta}(a + jh)) = \tilde{v}_j \quad \text{for } 1 \leq j \leq 4.$$

For $1 \leq j \leq 4$, this only determines the $\tilde{\theta}_j := \tilde{\theta}(a + jh)$ modulo 2π . To encourage simpler elastica the $\tilde{\theta}_j$ are chosen⁶ for $j = 1, 2, 3, 4$, as close as possible to $\tilde{\theta}_{j-1}$ consistent with (13). Then we interpolate accordingly⁷. A reasonable choice for $\tilde{\theta} : [a, b] \rightarrow \mathbb{R}$ is the natural cubic spline⁸ satisfying $\tilde{\theta}(a + jh) = \tilde{\theta}_j$ for $0 \leq j \leq 4$.

In short, Step 1 proceeds as follows:

- (1) If necessary, translate and rotate the data so that $x_a = (0, 0)$ and $v_a = (0, 1)$. Set $h := (b - a)/4$, $\tilde{v}_0 := v_a$, $\tilde{v}_4 := v_b$.
- (2) Define \tilde{w}_2 is given by formula (9).
- (3) Define $\tilde{w}_{1,3}$ by formula (10), then \tilde{w}_1 and \tilde{w}_3 by formula (12).
- (4) Set $\tilde{v}_j := \tilde{w}_j/\|\tilde{w}_j\|$ for $j=1,2,3$.
- (5) Set $\tilde{\theta}_0 := 0$ and, for $1 \leq j \leq 4$, choose $\tilde{\theta}_j$ so that $(\cos \tilde{\theta}_j, \sin \tilde{\theta}_j) = \tilde{v}_j$ with the $|\tilde{\theta}_j - \tilde{\theta}_{j-1}|$ as small as possible.
- (6) Let $\tilde{\theta} : [a, b] \rightarrow \mathbb{R}$ be the natural cubic spline satisfying $\tilde{\theta}(a + jh) = \tilde{\theta}_j$ for $0 \leq j \leq 4$.

Although $\tilde{\theta}$ approximates θ with at most $O(h^3)$ errors, the actual bounds on the $v^{(m)}$ may be difficult to estimate. So in practice it may be hard to say exactly how good the approximation is. Our algorithm is intended for relatively uncomplicated elastica x . So it is interesting to compare θ and $\tilde{\theta}$ in Example 1 which, as seen in Example 2, is a nontrivial case.

Example 3. *In Figure 3, the graph of $\tilde{\theta}$ (yellow) is not a highly accurate estimate of the graph of θ (blue) for Example 1. On the other hand, there do not seem to be any very remarkable differences between the two curves: this is all that is needed to begin the second step of our algorithm. It takes 56 seconds to plot the graph of θ by integrating the known closed-form solution for κ , compared with 0.197 seconds for plotting $\tilde{\theta}$. So solving the initial value problem from the closed-form solution is already time-consuming. Our algorithm, whose second step is given in Section 5, solves the much harder boundary-value problem. \square*

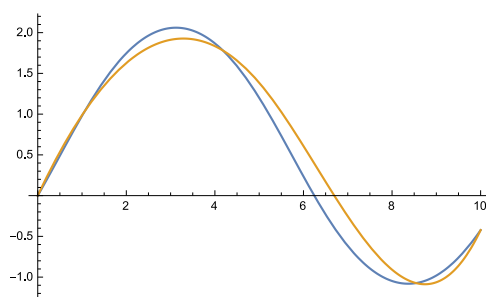


FIGURE 3. Liftings θ (blue) and $\tilde{\theta}$ (yellow) in Example 3.

⁵Other kinds of estimates can also be made, but this has the virtue of simplicity.

⁶This corresponds to smallest total curvatures over the $[a + (j - 1)h, a + jh]$ for $j = 1, 2, 3, 4$.

⁷The specific method of interpolation is not especially critical. Much more significantly, our construction of $\tilde{\theta}$ discourages exotic solutions of the subsequent optimisation problem.

⁸This corresponds to minimising the L^2 norm of κ , at least for the initial guess.

5. STEP 2

Step 1 gives a lifting $\tilde{\theta}$ of some uncomplicated curve \tilde{x} that approximately satisfies the end conditions. Next we approximately minimise the elastic energy, and more nearly satisfy the end conditions. This is done as follows.

- (1) For an integer n greater than 4, redefine $h := (b - a)/n$ and evaluate $\tilde{\theta}$ at $a + jh$ for $0 \leq j \leq n$ to give $\phi_{(0)} \in \mathbb{R}^{n+1}$.
- (2) For variable⁹ $\phi \in \mathbb{R}^{n+1}$, approximate $\int_a^b v(t)dt \in E^2$ by a sum of the form $S(\phi) := \sum_{j=0}^n q_j (\cos \phi_j, \sin \phi_j) \in E^2$ for suitable constant¹⁰ $q = (q_0, q_1, \dots, q_n) \in \mathbb{R}^{n+1}$.
- (3) Starting with $\phi_{(0)}$ as an initial guess, numerically minimise $\sum_{j=1}^n (\phi_j - \phi_{j-1})^2$ with $\phi_0 = \theta_a$, subject to the trigonometric constraints $S(\phi) = x_b - x_a$.
- (4) Interpolate the¹¹ minimiser ϕ by some convenient¹² C^2 curve $\hat{\theta} : [a, b] \rightarrow \mathbb{R}$.
- (5) Setting $\hat{v}(t) := (\cos \hat{\theta}(t), \sin \hat{\theta}(t))$, take \hat{x} to be a numerical solution¹³ of $\dot{\hat{x}}(t) = \hat{v}(t)$ with $\hat{x}(a) = x_a$.

There is another option, namely to start Step 2 with n small, then gradually increase n , repeating Step 2 with initial estimates of θ from previous optimisations. This more gradual movement from \tilde{x} to \hat{x} might occasionally be advantageous, but we have had no difficulty with the method as it stands. As explained in §3 and illustrated in §6, it seems reasonable to hope that the present method for finding clamped elastica is more robust, gives better results, and is faster than standard discretisation.

6. COMPARISONS WITH STANDARD DISCRETISATION

Example 4. With boundary conditions from Example 2 and $n = 20$, the estimate $\hat{x} : [a, b] \rightarrow E^2$ is almost¹⁴ indistinguishable from the original elastica x in Figure 2. It takes 0.066 seconds to compute \hat{x} , compared with 597 seconds for the standard discretisation (dotted). The discrete energy of \hat{x} is 0.0425, compared with 0.54 for the standard discretisation. \square

Example 5. Taking $b = 15$ instead of 10 in Example 4, we have $x_b = (4.38081, 6.00329)$, $v_b = (-0.0106571, 0.999943)$, the standard discretisation (red in Figure 3) takes 288.54 seconds. The standard discretisation has energy 0.0968 compared with 0.1465 for the original elastica x and, correspondingly, has a somewhat simpler appearance. So, on this occasion, the standard discretisation is preferable¹⁵ to x . Our method is better still, taking 0.054 seconds to compute \hat{x} . The even less complicated appearance of \hat{x} (continuous red) in Figure 3 is consistent with its lower discrete energy of 0.0292. \square

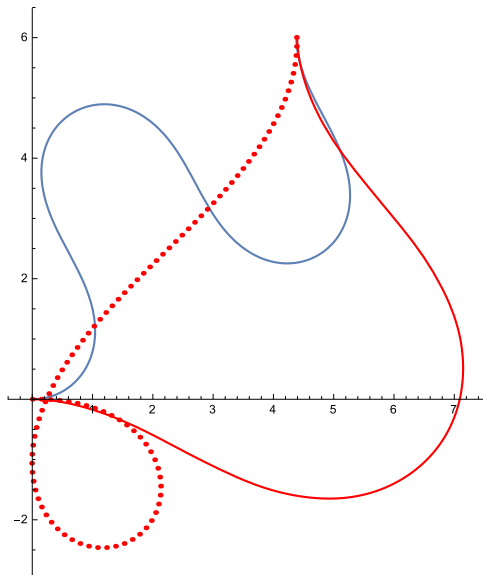


FIGURE 4. x (blue), standard discretisation (red dotted) and \hat{x} (continuous red) in Example 5.

Example 6. Increasing b to 20, x becomes more convoluted with discrete energy 0.351. Using boundary data from x , Mathematica takes 778 seconds to report failure of standard discretisation (nonconvergence, unusable output). So there is some question about robustness of standard discretisation, at least when pushed to this extent. On the other hand, our algorithm for finding \hat{x} takes 0.122 seconds. Consistent with its uncomplicated appearance (red in Figure 6), \hat{x} has discrete energy 0.0758, which compares well with the energy of x . \square

⁹corresponding notionally to an unknown lifting θ of v for the unknown uncomplicated elastica x

¹⁰When n is even, the Composite Simpson's Rule may be used, namely $q = h(1, 4, 2, 4, 2, 4, \dots, 2, 4, 1)/3$.

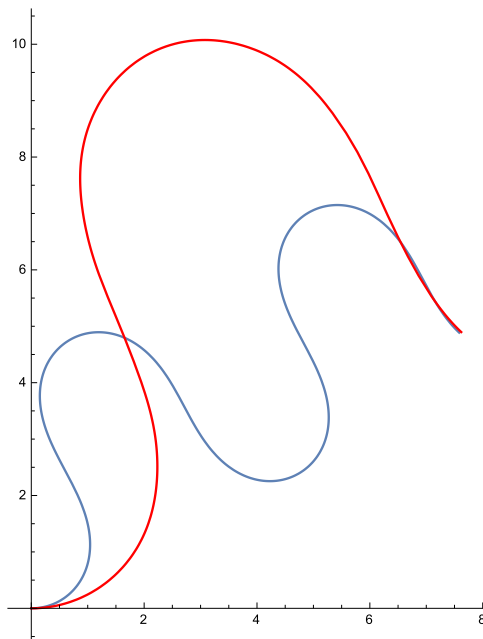
¹¹we hope, but possibly local minimiser

¹²such as the natural cubic spline

¹³Mathematica's *NDSolve* may be used.

¹⁴With $n = 40$ there is no visible difference at all.

¹⁵The aim is not to recover x , rather to minimise elastic energy subject to the given length and boundary conditions (in this case read from x).

FIGURE 5. x (blue) and \hat{x} (red) in Example 6.

7. CONCLUSION

A method is given for estimating clamped plane elastica. Arguments are made, and evidence is provided by way of illustrative examples, suggesting that the new method is quicker and more robust than standard discretisation, and more likely to give elastica of low energy. Just as for standard discretisation, no use is made of the known solutions for elastica in terms of elliptic functions. An extension to calculating general elastic splines is kept for a future paper.

REFERENCES

- [1] DAntonio L. The fabric of the universe is most perfect: Eulers research on elastic curves, in Euler at, 300: an appreciation, 239–260, Mathematical Association of America (2007).
- [2] Bruckstein, Alfred M., Holta Robert, J., and Netravalia, Arun, N., Discrete elastica, *Applicable Analysis* 78 (2001) 453–485.
- [3] Brunnett, Guido and Wendt, Jörg, A univariate method for plane elastic curves, *Computer Aided Geometric Design* 14 (1997) 273–292.
- [4] Edwards, J.A., Exact equations of the nonlinear spline, *ACM Transactions on Mathematical Software* 18 (1992), 174–192.
- [5] Levien, Raph, The elastica: a mathematical history, Technical Report No. UCB/EECS-2008-103 <http://www.eecs.berkeley.edu/Pubs/TechRpts/2008/EECS-2008-103.html> August 23, 2008.
- [6] Pontryagin, L.S., Boltyanskii, Gamkrelidze, R.V., Mischenko, E.F., “TheMathematical Theory of Optimal Processes,” (transl. K.N. Tiroogoff, ed. L.W. Neustadt), Gordon and Breach 1986.
- [7] Singer, David A., Lectures on Elastic Curves and Rods, in AIP Conference Proceedings 1002, Curvature and Variational Modelling in Physics and Biophysics, Santiago de Compostela, Spain, 17–18 September 2007 (eds Óscar J. Garay, Eduardo Garcia-Río, Ramón Vázquez-Lorenzo), 3–32.

LYLE.NOAKES@UWA.EDU.AU (SCHOOL OF MATHEMATICS & STATISTICS, THE UNIVERSITY OF WESTERN AUSTRALIA, 35 STIRLING HIGHWAY, CRAWLEY, WA 6009, AUSTRALIA)